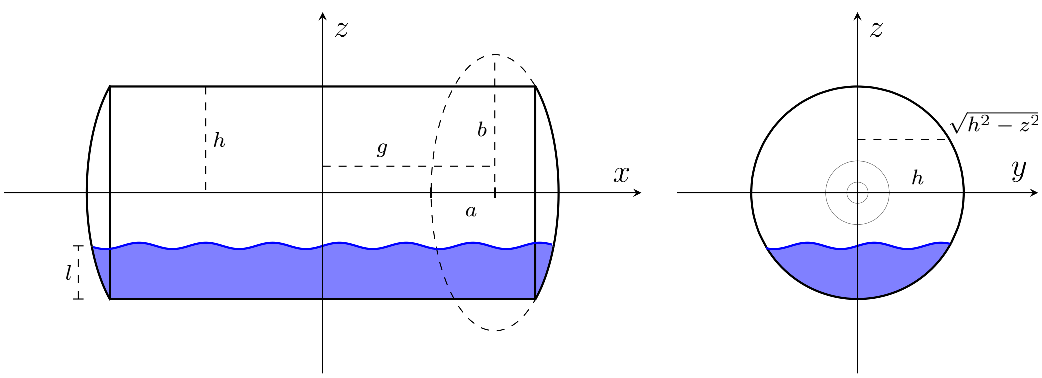
Content Volume of Cylindrical Tanks with Spheroid Segment Walls[1]

We consider a tank with walls given by segments of spheroids. In order to parameterize the outline of the tank, we fix a coordinate system with the x-axis identical to the common axis of the cylinder, like in the following illustration:



(a) side view   (b) front view

With this coordinate system and the notations from the illustration, the walls of the tank are given by the equations

$$\frac{(x \pm g)^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1.$$

We conclude, that the x-coordinates of the tank interior are confined by

$$-g - \frac{a}{b} \cdot \sqrt{b^2 - y^2 - z^2} \leq x \leq g + \frac{a}{b} \cdot \sqrt{b^2 - y^2 - z^2}.$$

It follows, that the volume V of the tank content with filling level l is given by the following integral:

$$\begin{aligned} V &= \int_{-h}^{l-h} \int_{-\sqrt{h^2-z^2}}^{\sqrt{h^2-z^2}} \int_{-\frac{a}{b}\sqrt{b^2-z^2-y^2}-g}^{\frac{a}{b}\sqrt{1-z^2-y^2}+g} 1 \; \mathrm{d}x \; \mathrm{d}y \; \mathrm{d}z \\ &= 4\,g \int_{-h}^{l-h} \sqrt{h^2 - z^2} \; \mathrm{d}z + \frac{2a}{b} \int_{-h}^{l-h} \int_{-\sqrt{h^2-z^2}}^{\sqrt{h^2-z^2}} \sqrt{b^2 - z^2 - y^2} \; \mathrm{d}y \; \mathrm{d}z. \end{aligned}$$

[1]Without warranty. I accept no responsibility if your tank breaks =) If you find an error in this calculation, please let me know (marc.diesse@hs-heilbronn.de).

We proceed with the well known antiderivative:

$$\int \sqrt{s^2 - x^2}\;\mathrm{d}x = \frac{s^2}{2}\,\arcsin\left(\frac{x}{s}\right) + \frac{x}{2}\,\sqrt{s^2 - x^2}, \quad s > 0,\ |x| < s.$$

So we have for V:

$$\begin{aligned} V = 2\,g\,h^2\,\arcsin\left(\frac{z}{h}\right) + 2\,g\,z\,\sqrt{h^2 - z^2}\,\Bigg|_{-h}^{l-h} &+ \frac{2a}{b}\int_{-h}^{l-h} \sqrt{b^2 - h^2}\,\sqrt{h^2 - z^2}\;\mathrm{d}z \\ &+ \frac{2a}{b}\int_{-h}^{l-h} (b^2 - z^2)\,\arcsin\left(\frac{\sqrt{h^2 - z^2}}{\sqrt{b^2 - z^2}}\right)\mathrm{d}z \\ = \left(2\,g + \frac{a}{b}\,\sqrt{b^2 - h^2}\right)\left(h^2\,\arcsin\left(\frac{z}{h}\right) + z\,\sqrt{h^2 - z^2}\right)\Bigg|_{-h}^{l-h} & \\ &+ \frac{2a}{b}\int_{-h}^{l-h} b^2\,\arcsin\left(\frac{\sqrt{h^2 - z^2}}{\sqrt{b^2 - z^2}}\right)\mathrm{d}z \\ &- \frac{2a}{b}\int_{-h}^{l-h} z^2\,\arcsin\left(\frac{\sqrt{h^2 - z^2}}{\sqrt{b^2 - z^2}}\right)\mathrm{d}z. \end{aligned}$$

We now investigate the three terms of V seperately:

First Term: With $q := \sqrt{b^2 - h^2}$ and $r := \frac{a}{b}$, we have

$$\begin{aligned} V_1 :=& (2\,g + r\,q)\,\left(h^2\,\arcsin\left(\frac{z}{h}\right) + z\,\sqrt{h^2 - z^2}\right)\Bigg|_{-h}^{l-h} \\ =& (2\,g + r\,q)\left(h^2\left(\arcsin\left(\frac{l-h}{h}\right) - \lim_{L\to -h}\arcsin\left(\frac{L}{h}\right)\right)\right. \\ & \left. + (l-h)\,\sqrt{h^2 - (l-h)^2}\right) \end{aligned}$$

And because of $\lim_{x\to -1}\arcsin(x) = -\pi/2$ we get:

$$V_1 = (2\,g + r\,q)\left(h^2\,\left(\arcsin\left(\frac{l-h}{h}\right) + \frac{\pi}{2}\right) + (l-h)\,\sqrt{2lh - l^2}\right). \tag{1}$$

Second Term: We have to solve the indefinite integral

$$I_1 := \int \arcsin\left(\frac{\sqrt{h^2 - z^2}}{\sqrt{b^2 - z^2}}\right)\mathrm{d}z,$$

for $b > h > |z|$. Partial Integration for $f_1 = 1,\ f_2 = \arcsin\left(\frac{\sqrt{h^2-z^2}}{\sqrt{b^2-z^2}}\right)$ gives:

$$I_1 = z\,\arcsin\left(\sqrt{\frac{h^2 - z^2}{b^2 - z^2}}\right) + q\int \frac{z^2}{\sqrt{h^2 - z^2}\cdot(b^2 - z^2)}\,\mathrm{d}z,$$

where $q := \sqrt{b^2 - h^2}$. We substitute $z = h \cdot \sin(u)$ one the right summand of I_1 and get:

$$I_2 := \int \frac{z^2}{\sqrt{h^2 - z^2} \cdot (b^2 - z^2)} \, \mathrm{d}z = \int \frac{h^3 \cdot \sin^2(u) \cdot \cos(u)}{h \cdot \sqrt{1 - \sin(u)^2} \cdot (b^2 - h^2 \cdot \sin^2(u))} \, \mathrm{d}u \, \Bigg|_{u = \arcsin\left(\frac{z}{h}\right)}$$

Rearranging and simplifying gives

$$I_2 = h^2 \int \frac{\sin^2(u)}{b^2 - h^2 \cdot \sin^2(u)} \, \mathrm{d}u = h^2 \int \frac{\tan^2(u)}{b^2 \cdot (1 + \tan^2(u)) - h^2 \, \tan^2(u)} \, \mathrm{d}u.$$

with the substituition $\tan(u) = w$, $\mathrm{d}u = \frac{1}{1+w^2} \, \mathrm{d}w$ we have

$$I_2 = h^2 \int \frac{w^2}{b^2 \, (1 + w^2) - h^2 \, w^2} \cdot \frac{1}{1 + w^2} \, \mathrm{d}w = h^2 \int \frac{b^2/h^2}{b^2 \, (1 + w^2) - h^2 \, w^2} - \frac{1/h^2}{1 + w^2} \, \mathrm{d}w,$$

where the last step follows with partial fractions. We continue

$$I_2 = - \int \frac{1}{1 + w^2} \, \mathrm{d}w + b^2 \int \frac{1}{b^2 \cdot \left(1 + \frac{b^2 - h^2}{b^2} \, w^2\right)} \, \mathrm{d}w.$$

So finally with $q := \sqrt{b^2 - h^2}$, we have:

$$I_2 = - \arctan(w) + \frac{b}{q} \cdot \arctan\left(\frac{q}{b} \cdot w\right).$$

With the back-substituitions $w = \tan(u)$, $u = \arcsin(\frac{z}{h})$, I_1 then evaluates to

$$I_1 = z \cdot \arcsin\left(\sqrt{\frac{h^2 - z^2}{b^2 - z^2}}\right) - q \, \arcsin\left(\frac{z}{h}\right) + b \, \arctan\left(\frac{q}{b} \cdot \frac{z}{\sqrt{h^2 - z^2}}\right),$$

where we used the identity $\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$, $|x| < 1$. Now we calculate for the second term:

$$\begin{aligned} V_2 :=& 2\, r \int_{-h}^{l-h} b^2 \, \arcsin\left(\frac{\sqrt{h^2 - z^2}}{\sqrt{b^2 - z^2}}\right) \, \mathrm{d}z \\ =& 2\, b^2\, r \Bigg((l - h) \cdot \arcsin\left(\sqrt{\frac{2\, lh - l^2}{b^2 - (l - h)^2}}\right) \\ & + b \left(\arctan\left(\frac{q}{b} \cdot \frac{l - h}{\sqrt{2\, lh - l^2}}\right) - \lim_{z \to -h} \arctan\left(\frac{q}{b} \cdot \frac{z}{\sqrt{h^2 - z^2}}\right)\right) \\ & - q \left(\arcsin\left(\frac{l - h}{h}\right) - \lim_{z \to -h} \arcsin\left(\frac{z}{h}\right)\right)\Bigg). \end{aligned}$$

It is $\lim_{x \to -1} \arcsin(x) = \lim_{x \to -\infty} \arctan(x) = -\pi/2$. So we get:

$$\begin{aligned} V_2 = 2\, r\, b^2 \Bigg(& (l - h) \cdot \arcsin\left(\sqrt{\frac{2\, lh - l^2}{b^2 - (l - h)^2}}\right) - q \left(\arcsin\left(\frac{l - h}{h}\right) + \frac{\pi}{2}\right) \\ & + b \left(\arctan\left(\frac{q}{b} \cdot \frac{l - h}{\sqrt{2\, lh - l^2}}\right) + \frac{\pi}{2}\right)\Bigg). \end{aligned} \tag{2}$$

Third Term Now we solve the indefinite integral

$$I_3 = \int z^2 \arcsin\left(\sqrt{\frac{h^2 - z^2}{b^2 - z^2}}\right) \mathrm{d}z,$$

Partial Integration like before brings us to

$$I_3 = \frac{1}{3} z^3 \arcsin\left(\frac{\sqrt{h^2 - z^2}}{\sqrt{b^2 - z^2}}\right) + \frac{1}{3} q \int \frac{z^4}{\sqrt{h^2 - z^2} \cdot (b^2 - z^2)},$$

with $q := \sqrt{b^2 - h^2}$. We substitute $z = h \sin(u)$ and rearrange once more:

$$\begin{aligned} I_4 &= h^4 \int \frac{\sin^4(u)}{b^2 - h^2 \cdot \sin^2(u)} \mathrm{d}u \Bigg|_{u=\arcsin\left(\frac{z}{h}\right)} \\ &= h^2 \int \frac{\tan^4(u)}{b^2 \cdot (1 + \tan^2(u))^2 - (1 + \tan^2(u)) \cdot \tan^2(u) \cdot h^2} \mathrm{d}u. \\ &= h^4 \int \frac{\tan^4(u)}{(1 + \tan^2(u)) \cdot (b^2 + (b^2 - h^2) \cdot \tan^2(u))} \mathrm{d}u. \end{aligned}$$

With the substitution $\tan(u) = w$, $\mathrm{d}u = \frac{1}{1+w^2} \mathrm{d}w$, we have now:

$$I_4 = h^4 \int \frac{w^4}{(1 + w^2)^2 \cdot (b^2 + (b^2 - h^2) \cdot w^2)} \mathrm{d}u.$$

and with partial fractions:

$$\begin{aligned} I_4 = \ & h^4 \cdot \int \frac{1}{h^2 \cdot (w^2 + 1)^2} \mathrm{d}w + h^4 \int \frac{b^4}{h^4 \cdot (b^2 + (b^2 - h^2) \cdot w^2)} \mathrm{d}w \\ & - h^4 \int \frac{b^2 + h^2}{h^4 \cdot (w^2 + 1)} \mathrm{d}w \\ = \ & h^2 \int \frac{1}{(w^2 + 1)^2} \mathrm{d}w + b^2 \int \frac{1}{1 + \left(\frac{q}{b} w\right)^2} \mathrm{d}w - (b^2 + h^2) \int \frac{1}{1 + w^2} \mathrm{d}w \end{aligned}$$

with $\int \frac{1}{(w^2+1)^2} = 1/2 \cdot (w/(1 + w^2) + \arctan(w))$, we finally have

$$I_4 = \frac{h^2}{2} \cdot \frac{w}{w^2 + 1} + \frac{h^2}{2} \arctan(w) + \frac{b^3}{q} \arctan\left(\frac{q}{b} w\right) - (b^2 + h^2) \arctan(w).$$

and with back-substitution:

$$
\begin{aligned}
I_4 &= \frac{h^2}{2} \cdot \frac{\tan(u)}{1+\tan(u)^2} + \frac{h^2}{2}\,u + \frac{b^3}{q}\ \arctan\left(\frac{q}{b}\tan(u)\right) - (b^2+h^2)\,u \\
&= \frac{h^2}{2}\sin(u)\cos(u) + \left(\frac{h^2}{2} - (b^2+h^2)\right) \cdot \arcsin\left(\frac{z}{h}\right) + \frac{b^3}{q} \cdot \arctan\left(\frac{q}{b} \cdot \frac{z}{\sqrt{h^2-z^2}}\right) \\
&= \frac{z}{2}\sqrt{h^2-z^2} - \left(\frac{2\,b^2+h^2}{2}\right) \cdot \arcsin\left(\frac{z}{h}\right) + \frac{b^3}{q} \cdot \arctan\left(\frac{q}{b} \cdot \frac{z}{\sqrt{h^2-z^2}}\right).
\end{aligned}
$$

So, putting it together:

$$
\begin{aligned}
I_3 = \frac{z^3}{3}\ \arcsin\left(\frac{\sqrt{h^2-z^2}}{\sqrt{b^2-z^2}}\right) + \frac{q\,z}{6}\sqrt{h^2-z^2} - q\left(\frac{2\,b^2+h^2}{6}\right) \cdot \arcsin\left(\frac{z}{h}\right) \\
+ \frac{b^3}{3} \cdot \arctan\left(\frac{q}{b} \cdot \frac{z}{\sqrt{h^2-z^2}}\right).
\end{aligned}
$$

Now we calculate for the third term:

$$
\begin{aligned}
V_3 :=& 2\,r\int_{-h}^{l-h} z^2\ \arcsin\left(\frac{\sqrt{h^2-z^2}}{\sqrt{b^2-z^2}}\right)\,\mathrm{d}z \\
=& \frac{r}{3}\Bigg(2\,(l-h)^3 \arcsin\left(\sqrt{\frac{2\,lh-l^2}{b^2-(l-h)^2}}\right) + q\,(l-h)\sqrt{2\,lh-l^2} \\
&- (2\,b^2+h^2)\,q\left(\arcsin\left(\frac{l-h}{h}\right) - \lim_{z\to-h}\arcsin\left(\frac{z}{h}\right)\right) \\
&+ 2\,b^3\left(\arctan\left(\frac{q}{b} \cdot \frac{l-h}{\sqrt{2\,lh-l^2}}\right) - \lim_{z\to-h}\arctan\left(\frac{q}{b} \cdot \frac{z}{\sqrt{h^2-z^2}}\right)\right)\Bigg).
\end{aligned}
$$

As before with $\lim_{x\to-1}\arcsin(x) = \lim_{x\to-\infty}\arctan(x) = -\pi/2$ we get:

$$
\begin{aligned}
V_3 = \frac{r}{3}\Bigg(2\,(l-h)^3 \cdot \arcsin\left(\sqrt{\frac{2\,lh-l^2}{b^2-(l-h)^2}}\right) + q\,(l-h)\sqrt{2\,lh-l^2} \\
- \left(2\,b^2+h^2\right)\,q\left(\arcsin\left(\frac{l-h}{h}\right) + \frac{\pi}{2}\right) \\
+ 2\,b^3\left(\arctan\left(\frac{q}{b} \cdot \frac{l-h}{\sqrt{2\,lh-l^2}}\right) + \frac{\pi}{2}\right)\Bigg).
\end{aligned} \tag{3}
$$

The Content Volume With V_1,V_2,V_3 as calculated in (1), (2), (3), we can now derive the following term for the Volume V:

$$\begin{aligned}
V =& V_1 + V_2 - V_3 \\
=& \left((2\,g + r\,q)\,h^2 - 2\,q\,r\,b^2 + \frac{q\,r\,(2\,b^2 + h^2)}{3} \right) \cdot \left(\arcsin\left(\frac{l-h}{h} \right) + \frac{\pi}{2} \right) \\
&+ \left((2g + r\,q)\,(l-h) - \frac{r\,q\,(l-h)}{3} \right) \cdot \sqrt{2\,lh - l^2} \\
&+ \left(2\,r\,b^2\,(l-h) - \frac{2\,r\,(l-h)^3}{3} \right) \cdot \arcsin\left(\sqrt{\frac{2\,lh - l^2}{b^2 - (l-h)^2}} \right) \\
&+ \left(2\,b^3\,r - \frac{2\,b^3\,r}{3} \right) \cdot \left(\arctan\left(\frac{q}{b} \cdot \frac{l-h}{\sqrt{2\,lh - l^2}} \right) + \frac{\pi}{2} \right),
\end{aligned}$$

which we simplify to

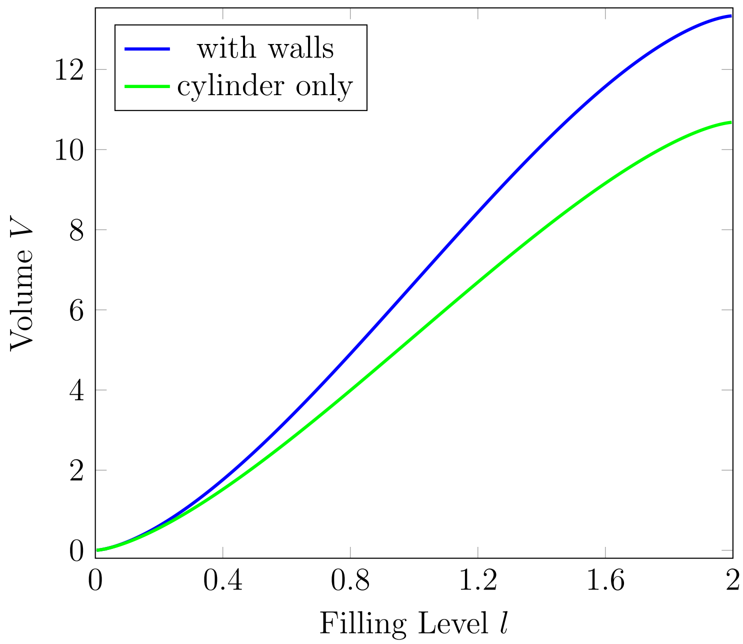
$$\begin{aligned}
V =& \left(2\,g\,h^2 + \frac{4\,q\,r\,\left(h^2 - b^2\right)}{3} \right) \cdot \left(\arcsin\left(\frac{l-h}{h} \right) + \frac{\pi}{2} \right) \\
&+ \left((l-h)\,\frac{6\,g + 2\,r\,q}{3} \right) \cdot \sqrt{2\,lh - l^2} \\
&+ \left(2\,(l-h)\,r\,\frac{3\,b^2 - (l-h)^2}{3} \right) \cdot \arcsin\left(\sqrt{\frac{2\,lh - l^2}{b^2 - (l-h)^2}} \right) \\
&+ \frac{4\,b^3\,r}{3} \cdot \left(\arctan\left(\frac{q}{b} \cdot \frac{l-h}{\sqrt{2\,lh - l^2}} \right) + \frac{\pi}{2} \right),
\end{aligned}$$

$$\text{with } r = \frac{a}{b} \text{ and } q = \sqrt{b^2 - h^2}.$$

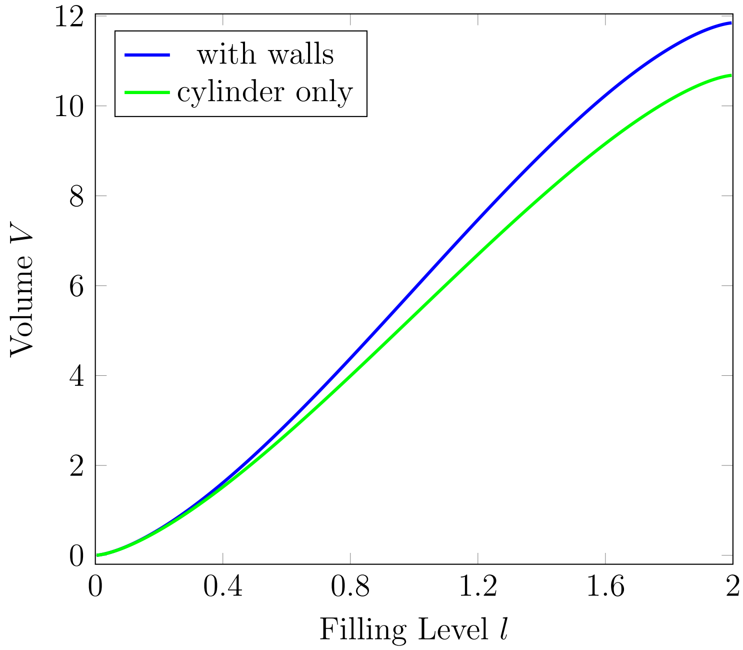
Spherical Walls In the special case of spherical walls with constant curvature K we have $r = \frac{a}{b} = 1$ and a radius of $b = 1/\sqrt{K}$, since $b > h$ we can have a maximal curvature of $1/h^2$, when the walls are complete hemispheres. If the curvature and the length d of the cylindric part is given, we calculate:

$$g = d - \sqrt{\frac{1}{K} - h^2}.$$

Content Volume Plots In the following figures we plot the content volume for tanks with different wall curvatures.



Content volume for a tank with $g = 1.7$, $h = 1$ and wall curvature $K = 0.9$



Content volume for a tank with $g = 1.7$, $h = 1$ and wall curvature $K = 0.4$