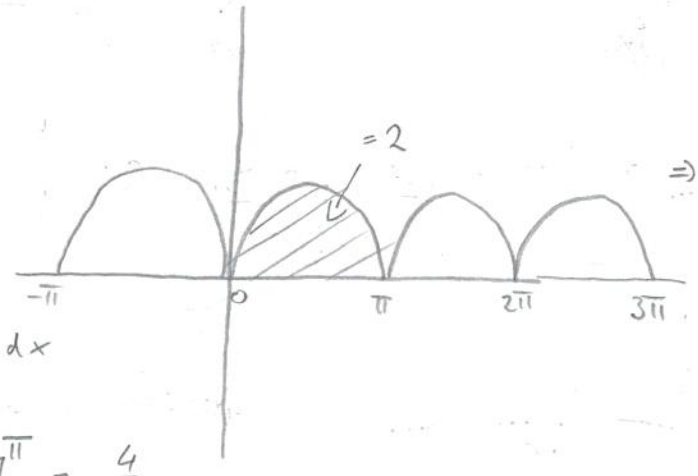
Musterrechnung Fourierkoeff.

• zu $f(x) = |\sin(x)|$



$p = \pi$

$\Rightarrow \omega = \frac{2\pi}{p} = 2$

$$a_0 = \frac{2}{p}\int_0^{\pi} |\sin(x)|\,dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\,dx$$

$$= -\frac{2}{\pi}\left[\cos(x)\right]_0^{\pi} = \underline{\underline{\frac{4}{\pi}}}$$

$b_k = 0$ (f gerade)

$$a_k,\ k \geq 0 = \frac{2}{p}\cdot\int_0^{\pi} |\sin(x)|\cdot\cos(2\cdot k\cdot x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(2k\cdot x)\,dx$$

<u>2 Wege:</u> 1. mit Additionstheoremen 2. mit partieller Integration

<u>1.</u>

$$\sin(x+2kx) = \sin(x)\cos(2kx) + \cos(x)\sin(2kx)$$
$$\sin(x-2kx) = \sin(x)\cos(2kx) - \cos(x)\sin(2kx)$$

$$\Rightarrow \sin(x)\cos(2kx) = \frac{1}{2}\left(\sin(x+2kx) + \sin(x-2kx)\right)$$

Also $a_k = \frac{2}{\pi}\cdot\frac{1}{2}\cdot\int_0^{\pi} \sin(x(1+2k)) + \sin(x(1-2k))\,dx$

$$= \frac{1}{\pi}\cdot\left(\int_0^{\pi}\sin(x(1+2k))\,dx + \int_0^{\pi}\sin(x(1-2k))\,dx\right) \underset{\substack{1+2k\neq 0\\ 1-2k\neq 0}}{=} \frac{1}{\pi}\left(\left[-\frac{1}{1+2k}\cos(x(1+2k))\right]_0^{\pi} + \left[-\frac{1}{1-2k}\cos(x(1-2k))\right]_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left(-\frac{1}{1+2k}\underbrace{\left[\cos(x(1+2k))\right]_0^{\pi}}_{\underbrace{\cos(\pi(1+2k))}_{\underbrace{\cos(\pi+2k\pi)}_{-1}} - \underbrace{\cos(0)}_{1}} - \frac{1}{1-2k}\underbrace{\left[\cos(x(1-2k))\right]_0^{\pi}}_{\underbrace{\cos(\pi(1-2k))}_{\underbrace{\cos(\pi-2k\pi)}_{-1}} - \underbrace{\cos(0)}_{1}}\right)$$

$$= \frac{1}{\pi}\left(-\frac{1}{1+2k}(-1-1) - \frac{1}{1-2k}(-1-1)\right) = \frac{2}{\pi}\left(\frac{1}{1+2k} + \frac{1}{1-2k}\right) = \underline{\underline{\frac{2}{\pi}\left(\frac{2}{1-4k^2}\right)}}$$

2.
$$\int_0^{\pi} \sin(x)\cos(2kx)\,dx = \cos(2kx)\cdot(-\cos(x))\Big|_0^{\pi} - \int_0^{\pi} 2k\cdot\cos(x)\sin(2kx)\,dx$$

$$= \underbrace{\cos(2k\pi)\cdot(-\cos(\pi))}_{1} - \underbrace{\cos(0)\cdot(-\cos(0))}_{-1} - 2k\int_0^{\pi}\cos(x)\sin(2kx)\,dx$$

$$= 2 - 2k\left(\underbrace{\sin(2kx)\cdot\sin(x)\Big|_0^{\pi}}_{0} - 2k\int_0^{\pi}\sin(x)\cos(2kx)\,dx\right)$$

$$= 2 - 2k\cdot\left(-2k\int_0^{\pi}\sin(x)\cos(2kx)\,dx\right) = 2 + 4k^2\cdot\int_0^{\pi}\sin(x)\cos(2kx)\,dx$$

$$\Rightarrow (1-4k^2)\cdot\int_0^{\pi}\sin(x)\cos(2kx)\,dx = 2$$

$$\underset{1-4k^2 \neq 0}{\Rightarrow} \int_0^{\pi}\sin(x)\cos(2kx)\,dx = \frac{2}{1-4k^2}$$

$$\Rightarrow a_k = \frac{2}{\pi}\cdot\int_0^{\pi}\sin(x)\cos(2kx)\,dx = \frac{2}{\pi}\left(\frac{2}{1-4k^2}\right)$$

Vorteil von 2. Man muss Additionstheoreme nicht auswendig wissen und Auswertung von sin und cos i.A. leichter.
Aber: mehr Rechenarbeit

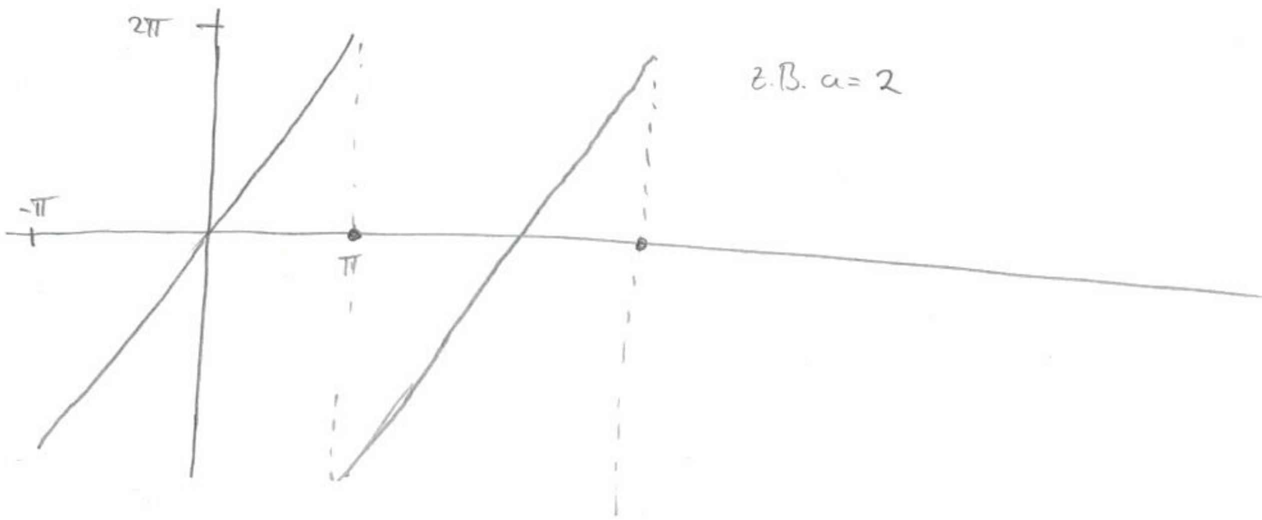
Ergebnis

$$|\sin(x)| = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(\omega kx) + b_k\sin(\omega kx)$$

$$= \frac{2}{\pi} + \sum_{k=1}^{\infty}\frac{2}{\pi}\left(\frac{2}{1-4k^2}\right)\cos(2kx)$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{1-4k^2}\cos(2kx)$$

• zu $f_0(x) = a \cdot x$ auf $(-\pi, \pi)$, periodisch fortgesetzt
$a \in \mathbb{R} \setminus \{0\}$ mit $f(x) = 0$, $x \in \{\pi + 2k\pi \mid k \in \mathbb{Z}\}$



$p = 2\pi \Rightarrow \omega = 1$ Funktion ungerade $\Rightarrow a_k = 0 \quad k \geq 0$

$$\underset{\substack{\uparrow \\ f \text{ ungerade}}}{b_k} = \frac{4}{p} \cdot \int_0^{\pi} ax \cdot \sin(kx)\,dx = \frac{4a}{2\pi} \cdot \int_0^{\pi} \underset{\downarrow}{x} \cdot \underset{\uparrow}{\sin(kx)}\,dx$$

$$\overset{p.I}{=} \frac{2a}{\pi}\left(x \cdot \cos(kx) \cdot \frac{-1}{k} \Big|_0^{\pi} + \frac{1}{k}\int_0^{\pi} \cos(kx)\,dx \right)$$

$$= \frac{2a}{\pi}\left(\pi \cdot \underbrace{\cos(k\pi)}_{(-1)^k} \cdot \frac{-1}{k} + \frac{1}{k^2} \cdot \underbrace{\left[\sin(kx)\right]_0^{\pi}}_{0} \right)$$

$$= \underline{\underline{-\frac{2a}{k} \cdot (-1)^k}}$$

$$\Rightarrow f(x) = -\sum_{k=1}^{\infty} \frac{2a}{k} (-1)^k \sin(kx)$$

z.B. für $a = \frac{1}{\pi} \Rightarrow f(x) = -\sum_{k=1}^{\infty} \frac{2}{\pi k} (-1)^k \sin(kx)$