Aufgabe 1

(a) $f(x) = x^4 - 2x^2$

$f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$

$f''(x) = 12x^2 - 4$

Extremwerte $f'(x) = 0 \iff 4x(x-1)(x+1) = 0$

$\iff x = 0, 1, -1$

$f''(0) = -4 \Rightarrow$ lok. Max

$f''(1) = f''(-1) = 8 \Rightarrow$ lok Min

$(0, 0)$ lok Max

$(-1, -1)$ lok Min

$(1, -1)$ lok Min

(b) $f(x) = 2x^3 - 3x^2 + 4$

$f'(x) = 6x^2 - 6x = 6 \cdot x(x-1)$ $f''(x) = 12x - 6$

Extremwerte $f'(x) = 0 \iff 6x(x-1) = 0$

$\iff x = 0, 1$

$f''(0) = -6 \Rightarrow$ lok. Max $(0, 4)$

$f''(1) = 6 \Rightarrow$ lok Min $(1, 3)$

(c) $f(x) = e^{-x^2-2x+1}$

$f'(x) = e^{-x^2-2x+1} \cdot (-2x-2)$

$f''(x) = e^{-x^2-2x+1} \cdot (-2x-2)^2 + e^{-x^2-2x+1} (-2)$

$= e^{-x^2-2x+1} \left(4x^2+8x+4-2\right)$

$= e^{-x^2-2x+1} \left(4x^2+8x+2\right)$

Extremwerte

$f'(x) = 0 \iff e^{-x^2-2x+1} \cdot (-2x-2) = 0$

$\iff -2x-2 = 0$

$\iff -2 = 2x \iff x = -1$

$f''(-1) = \underbrace{e^{\cdots}}_{>0} \cdot (4-8+2) = e^{\cdots} \cdot (-2) < 0$

$\Rightarrow$ lok. Max $(-1, e^2)$

(d) $f(x) = 16 \cdot \cos(\pi/4 \cdot x) \quad -5 \le x \le 5$

$f'(x) = 16 \cdot \pi/4 \cdot (-\sin(\pi/4 x)) = -4\pi \cdot \sin(\pi/4 x)$

$f''(x) = -4\pi \cdot \pi/4 \cdot \cos(\pi/4 x) = -\pi^2 \cdot \cos(\pi/4 x)$

Extremwerte $f'(x) = 0 \iff -4\pi \cdot \sin(\pi/4\, x) = 0$

$\iff \sin(\pi/4\, x) = 0$

$\iff \pi/4\, x = k \cdot \pi, \ k \in \mathbb{Z}$

$\iff x = -4, 0, 4$

$f''(-4) = -\pi^2 \cdot \cos(-\pi) = -\pi^2 (-1) = \pi^2 > 0$

$\Rightarrow$ lok Min bei $(-4, -16)$

$f''(0) = -\pi^2 \cdot \cos(0) = -\pi^2 \cdot 1 < 0$

$\Rightarrow$ lok Max bei $(0, 16)$

$f''(4) = -\pi \cdot \cos(\pi) = -\pi^2 \cdot (-1) = \pi^2 > 0$

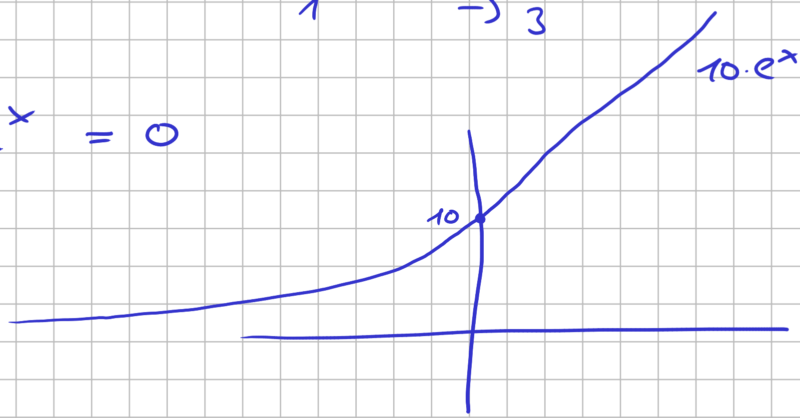
$\Rightarrow$ lok. Min bei $(4, -16)$

Aufgabe 2

(a) $\lim\limits_{x \to \infty} \frac{x^2+1}{x} = \lim\limits_{x \to \infty} \underbrace{\frac{x^2}{x}}_{\to \infty} \cdot \underbrace{\frac{1+\frac{1}{x^2}}{1}}_{\to 1} = \infty$

(b) $\lim\limits_{x \to -\infty} \frac{3x^2+x}{x^2-2} = \lim\limits_{x \to -\infty} \underbrace{\frac{x^2}{x^2}}_{=1} \cdot \underbrace{\frac{3+\frac{1}{x}}{1-\frac{2}{x^2}}}_{\to 3} = 3$

(c) $\lim\limits_{x \to -\infty} 10 \cdot e^x = 0$



(d) $\lim\limits_{x \to \infty} \frac{e^x}{x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim\limits_{x \to \infty} \frac{e^x}{1} = \infty$

(e) $\lim\limits_{x \to 0} \frac{\sin(x) - x}{x^2} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x \to 0} \frac{\cos(x) - 1}{2x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{x \to 0} \frac{\overset{\to 0}{-\sin(x)}}{2}$

$= 0$

f) $\lim_{x \to 0} x \cdot \ln(x) = \lim_{x \to 0} \frac{\ln(x)}{\frac{1}{x}}$

$\overset{"\frac{-\infty}{\infty}"}{=} \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} -\frac{1}{x} \cdot \frac{x^2}{1} = \lim_{x \to 0} -x$

$= 0$

Aufgabe 3

(a) $f(x) = x^3 - 6x^2 + 8$

$f'(x) = 3x^2 - 12x$

$f''(x) = 6x - 12$

Extremwerte

$f'(x) = 0 \iff 3x^2 - 12x = 0 \iff x(3x - 12) = 0$

$\iff x = 0, 4$

$f''(0) = -12 < 0 \implies$ lok Max bei $(0, 8)$

$f''(4) = 12 > 0 \implies$ lok Min bei $(4, -24)$

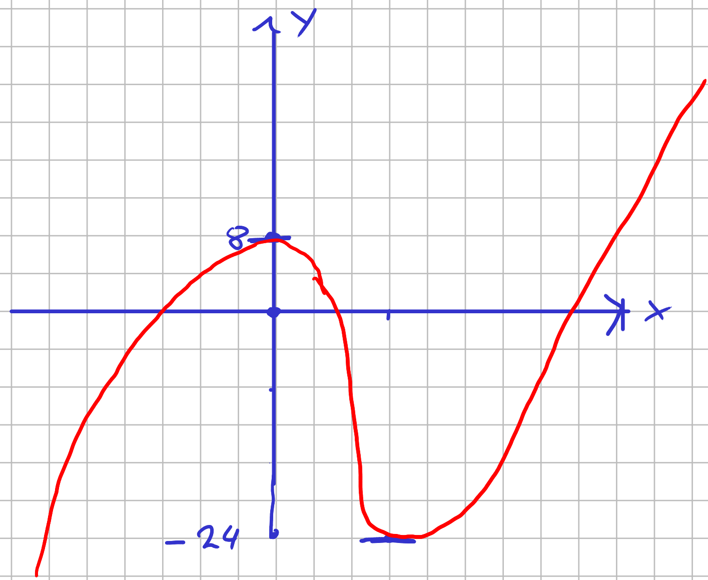
Keine Symmetrie

Nullstellen berechnen schwierig

Verhalten für $x \to \pm\infty$

$\lim_{x \to \infty} x^3 - 6x^2 + 8 = \infty$

$\lim_{x \to -\infty} x^3 - 6x^2 + 8 = -\infty$



(b) $\frac{x^2}{x+1} = f(x)$ $f'(x) = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{2x^2+2x-x^2}{(x+1)^2}$

$= \frac{x^2+2x}{(x+1)^2}$

$f''(x) = \frac{(2x+2)(x+1)^2 - (x^2+2x)\cdot 2\cdot(x+1)}{(x+1)^4}$

$= \frac{(2x+2)(x^2+2x+1) - (x^2+2x)(2x+2)}{(x+1)^4}$

$= \frac{2x^3+4x^2+2x+2x^2+4x+2 - (2x^3+2x^2+4x^2+4x)}{(x+1)^4} = \frac{2x+2}{(x+1)^4}$

Nst. $f(x)=0 \iff x=0$

Schnitt mit y-Achse $f(0)=0$

$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = -\infty$

Pol $x=-1$

$\lim_{x\to -1^-} f(x) = -\infty$

$\lim_{x\to -1^+} f(x) = \infty$

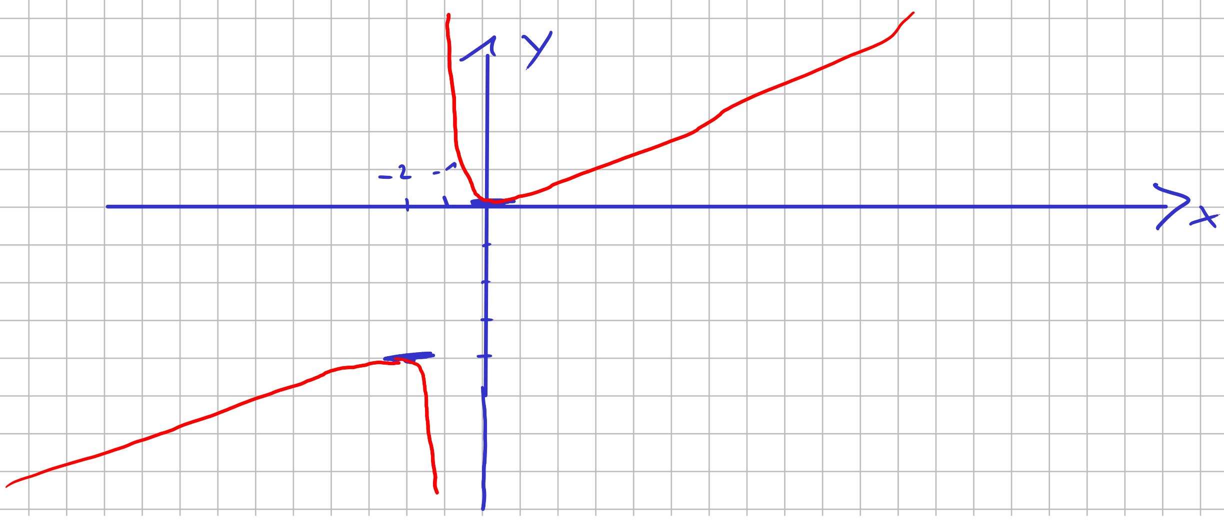
Extremwerte

$f'(x)=0 \iff x^2+2x = 0 \iff x(x+2)=0$

$\iff x = 0, -2$

$f''(0) = 2 > 0 \Rightarrow$ lok Min bei $(0,0)$

$f''(-2) = -2 < 0 \Rightarrow$ lok Max bei $(-2,-4)$



(c) $f(x) = (x+1)\cdot e^{-x}$

$$f'(x) = e^{-x} - (x+1)\, e^{-x} = e^{-x}\left(1-(x+1)\right)$$
$$= -xe^{-x}$$

$$f''(x) = -e^{-x} + xe^{-x} = e^{-x}(x-1)$$

Extremwerte $f'(x) = 0 \iff x = 0$

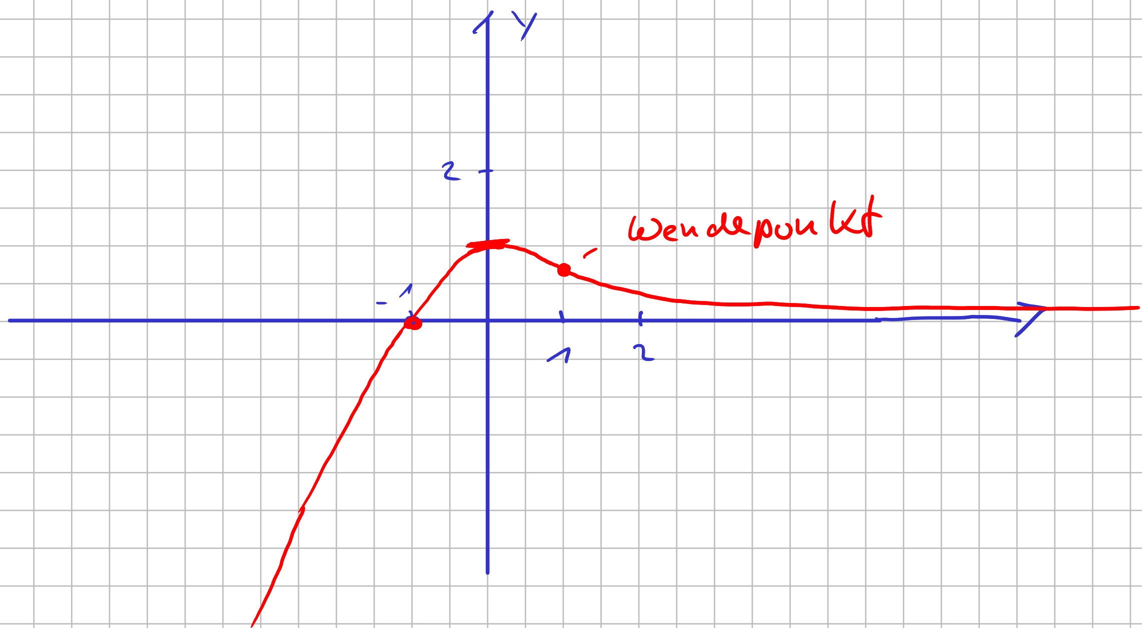
$$f''(0) = -1 \Rightarrow \text{lok Max bei } (0,1)$$

Verhalten für $x \to \pm\infty$

$$\lim_{x\to\infty} (x+1)\cdot e^{-x} = \lim_{x\to\infty} \frac{x+1}{e^x} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$$

$$\lim_{x\to-\infty} (x+1)\, e^{-x} = -\infty$$

Nullstelle bei $x = -1$



(d) $x \cdot \ln(x)$ $\quad \mathbb{D} = [0, \infty)$

$\lim_{x \to 0} x \cdot \ln(x) = 0$ s.o.

$f'(x) = \ln(x) + \frac{x}{x} = \ln(x) + 1$

$f''(x) = \frac{1}{x}$

$f'(x) = 0 \iff \ln(x) = -1 \iff x = e^{-1} = 1/e$

$\approx 0{,}37$

$f''(e^{-1}) = e > 0 \Rightarrow$ lok Min

bei $(e^{-1}, -e^{-1})$

Nst: $x \cdot \ln(x) = 0 \iff x = 1$

$\lim_{x \to \infty} x \cdot \ln(x) = \infty$

Wendepunkte $\quad f''(x) = 0 \iff \frac{1}{x} = 0$ ↯

$\Rightarrow$ keine Wendepkte.